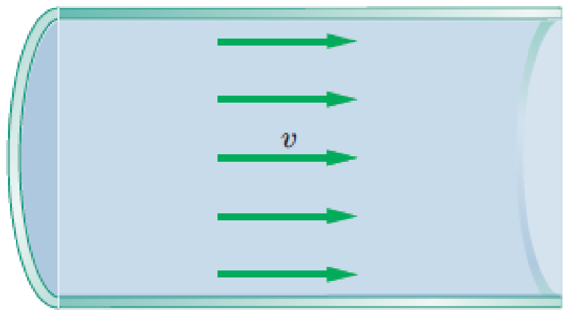
Sections 10.11 & 10.12: Poiseuille's equation

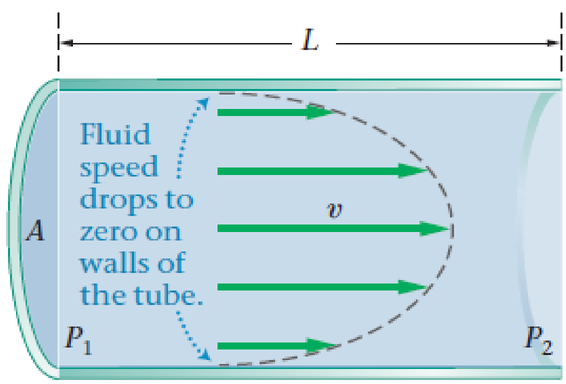
1. So far we have considered only "ideal" fluids. In particular, we've assumed that fluids flow without frictional losses and that the molecules in a fluid have no interaction with one another. In this section, we consider the consequences that follow from relaxing these assumptions.

2. When a block slides across a rough floor, it experiences a frictional force opposing the motion. Similarly, a fluid flowing past a stationary surface experiences a force opposing the flow. This tendency to resist flow is referred to as the **viscosity** of a fluid. Fluids like air have low viscosities, thicker fluids like water and blood are more viscous, and fluids like honey and motor oil are characterized by high viscosity. **Viscosity in a fluid is similar to friction between two solid surfaces.**

3. To be specific, consider a situation of great practical importance—the flow of a fluid through a tube. Examples of this type of system include water flowing through a metal pipe in a house and **blood flowing through an artery or a vein**. If the fluid were ideal, with zero viscosity, it would flow through the tube with a speed that is the same throughout the fluid, as indicated in **Figure (a)**. Real fluids with finite viscosity are found to have flow patterns like the one shown in **Figure (b)**. In this case, the fluid is at rest next to the walls of the tube and flows with its greatest speed in the center of the tube. Because adjacent portions of the fluid flow past one another with different speeds, a force must be exerted on the fluid to maintain the flow, just as a force is required to keep a block sliding across a rough surface.



(a)



(b)

4. The force causing a viscous fluid to flow is provided by the pressure difference, $P_1 - P_2$, across a given length, L, of tube (Figure b). Experiments show that the required pressure difference is proportional to the length of the tube and to the average speed, v, of the fluid. In addition, it is inversely proportional to the cross-sectional area, A, of the tube. Combining these observations, we can write the pressure difference in the following form:

$$P_1 - P_2 \propto \frac{vL}{A} \quad \text{(Eq 1)}$$

The constant of proportionality between the pressure difference and vL/A is 'related' to the **coefficient of viscosity**, $\boldsymbol{\eta}$, of a fluid. In fact, the proportionality constant is determined in such a way that the pressure difference is given by the following expression:

$$P_1 - P_2 = 8\pi\eta \frac{vL}{A} \quad \text{(Eq 2)}$$

From this equation we can see that the dimensions of the coefficient of viscosity are N . s/m^2. The SI unit for $\boldsymbol{\eta}$ is Pa . s (Pascal . second). A common non SI unit in the study of viscous fluids is the **poise** (P), named for the French physiologist Jean Leonard Marie Poiseuille (1797–1869) and defined as 1 poise = 1 P = 0.1 Pa . s. For example, the viscosity of water at room temperature (20°C) is 1.0 x 10^{-3} Pa . s and the viscosity of blood at 37 °C is 4 x 10^{-3} Pa . s. Some additional viscosities are given in **Table 10-3**.

5. A convenient way to characterize the flow of a fluid is in terms of its volume flow rate. Referring to Section 10-8 (**Equation 10-4b**), we see that the volume flow rate of a fluid is simply vA, where v is the average speed of the fluid and A is the cross-sectional area of the tube through which it flows. Solving (Eq 2) for the average speed gives $v = (P_1 - P_2)\, A/\, (8\pi\boldsymbol{\eta}L)$. Multiplying this result by the cross-sectional area of the tube yields the volume flow rate:

$$Q \equiv \frac{\Delta V}{\Delta t} = vA = \frac{(P_1 - P_2)A^2}{8\pi\eta L} \quad \text{(Eq 3)}$$

Using the fact that the cross-sectional area of a cylindrical tube is $A = \pi\, r^2$, where r is its radius, we obtain the result known as **Poiseuille's equation**:

$$Q \equiv \frac{\Delta V}{\Delta t} = \frac{(P_1 - P_2)\pi r^4}{8\eta L} \quad \text{(Eq 4} \equiv \text{Eq 10.9 of your text)}$$

Poiseuille's equation tells us that the flow rate is directly proportional to the pressure and it is inversely proportional to the viscosity of the fluid and the length of the tube. This is just what we might expect. Most significantly, though, that Q also depends on the **fourth power of the tube's radius**; thus a small change in radius corresponds to a large change in volume flow rate. To see the **significance of the r^4 dependence**, consider an artery that branches into an arteriole with half the artery's radius (see **Example 10-12**). Letting r go to $r/2$ in Poiseuille's equation, and solving for the pressure difference, we find

$$P_1 - P_2 = \frac{8\eta L}{\pi\left({r}/{2}\right)^4}\left(\frac{\Delta V}{\Delta t}\right) = 16\left[\frac{8\eta L}{\pi r^4}\left(\frac{\Delta V}{\Delta t}\right)\right] \text{ (Eq 5)}$$

Thus, the pressure difference across a given length of arteriole is **16 times** what it is across the same length of artery.
Similarly, a narrowing, or stenosis, of an artery (see Figure 10-32) or a cholesterol buildup can produce significant increases in blood pressure. For example, a reduction in radius of only 20%, from *r* to 0.8*r*, causes an increase in pressure by a factor of $(1/0.8)^4 \sim 2.4$. Thus, even a small narrowing of an artery can lead to an increased risk for heart disease and stroke.

6. EXAMPLE: BLOOD SPEED IN THE PULMONARY ARTERY

(a) The pulmonary artery, which connects the heart to the lungs, is 8.9 cm long and has a pressure difference over this length of 450 Pa (as shown in the figure below). If the inside radius of the artery is 2.3 mm, what is the average speed of blood in the pulmonary artery? **1.2 m/s**.
[Take the viscosity of the blood 2.72×10^{-3} Pa . s].
(b) Find the volume of blood that flows per second through the pulmonary artery. **20 cm^3**.
(c) If the radius of the artery is reduced by 18%, by what factor is the blood flow rate reduced? Assume that all other properties of the artery remain unchanged.
The flow rate is reduced by a factor of 1/0.45 = 2.2.