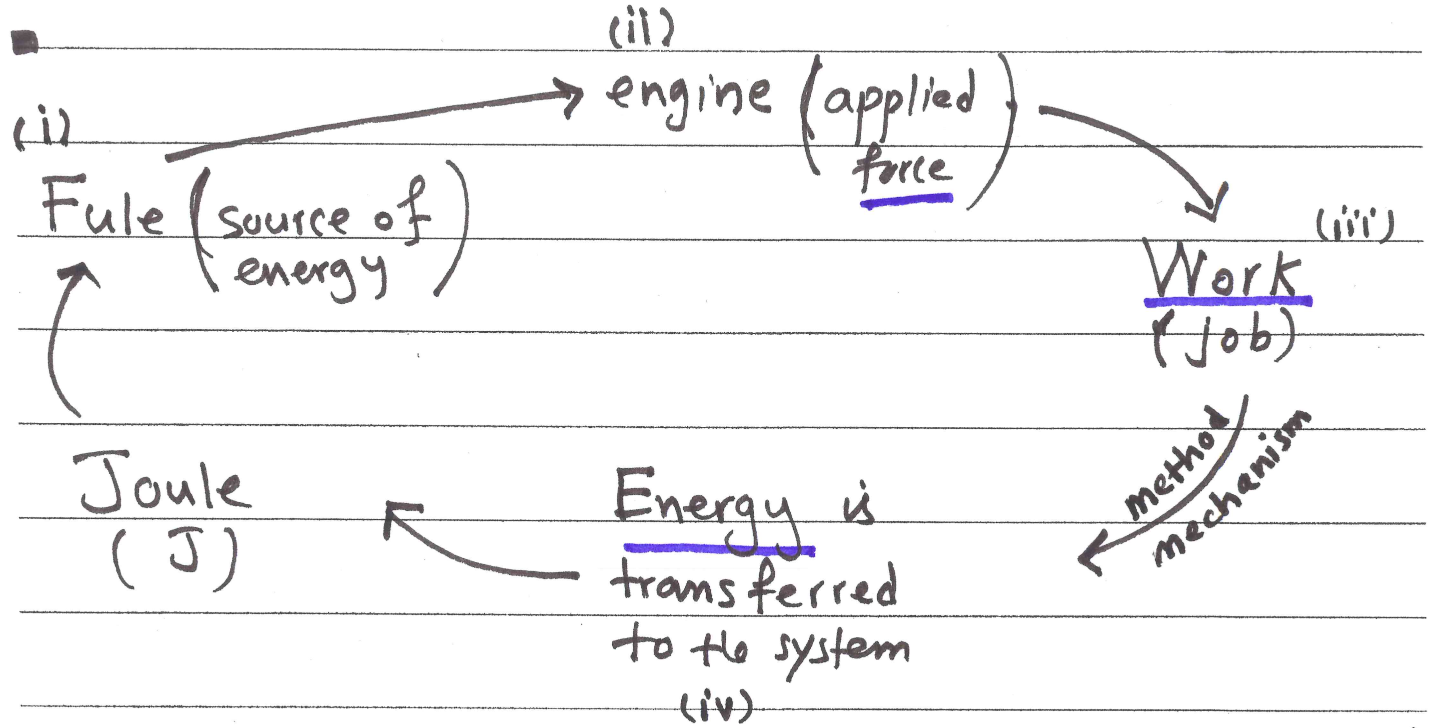
Chapter 6: Work-Energy Theorem

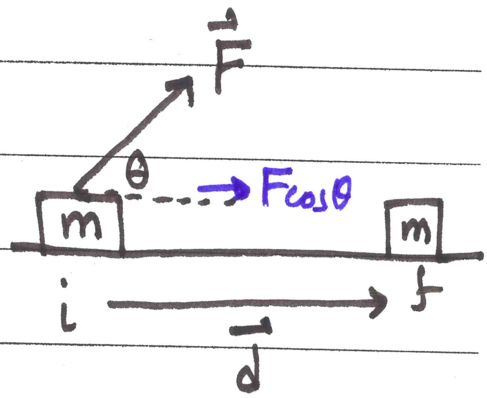


- Food → Stomach → Work (digest) → Energy (calorie)
- Wind → Wind turbine → Work → Energy (renewable)

⇒ Work is a method of transferring energy to an object or from an object as the result of the action of an external force. Positive work transfers energy <u>to</u> the object, and negative work transfers energy <u>from</u> the object.

- Work (W) done on an object is defined as the scalar product of the force vector acting upon the object and the displacement vector the object undegoes under the action of the force. By definition, work is a scalar. The SI unit for work is (N).(m) = Joule (J).
- Consider a block m pulled by a force $\vec{F}$ and displaced a displacement $\vec{d}$.

By definition, the work done by $\vec{F}$ on m is



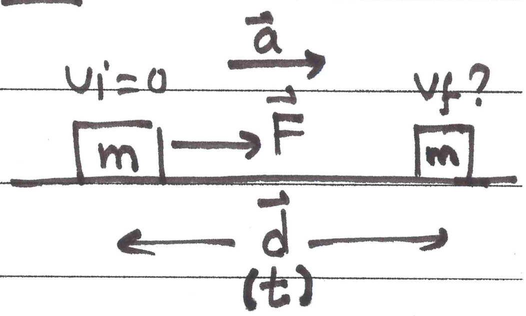
$$W_{\vec{F}} \equiv (F\cos\theta)(d)$$

$$\therefore \quad W_{\vec{F}} = |\vec{F}| * |\vec{d}| * \cos\theta . \qquad \text{eq (1)}$$

θ = angle between $\vec{F}$ and $\vec{d}$

- Notice that the work done by the gravitational force on m and the work done by the normal force on m are both zero because both forces are perpendicular to the displacement $\vec{d}$.

- One can understand the meaning of work as the influence of the force on the object. However, there is a radical contrast between the definition of work in physics and our everyday understanding of the term!
- <u>Problem 6.9</u>: (m) is accelerated from rest at constant rate, a, for (t) seconds. Find the <u>net</u> work on m.



"The net work done by the net force is equal to the algebraic sum of the work done by the individual forces". The mathematical reason for this lies in the distributive property of the scalar product. This fact is a major advantage of the work-energy approach because we do <u>NOT</u> need vector analysis!

(i) $\vec{F}_{net} = \vec{F} \Rightarrow W_{F_{net}} = |\vec{F}| * |\vec{d}|$. We know neither $\vec{F}$ nor $\vec{d}$!

$\Rightarrow F = ma$, $\boxed{d = \frac{at^2}{2}}$

$\therefore W_{F_{net}} = \left(\frac{m}{2}\right)(at)^2$ J.

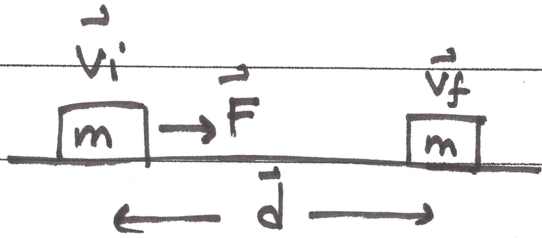
(ii) $W_{net} = W_{\vec{F}} + W_{\vec{g}} + W_{\vec{n}} = W_{\vec{F}} + 0 + 0 = \left(\frac{m}{2}\right)(at)^2 \, J.$

notice that the sum is algebraic; cheer up!

- Do example 6.1

⁂ Let's generalize problem 6.9 ⁂

The force $\vec{F}$ accelerates the block m from $\vec{v_i}$ to $\vec{v_f}$ over a displacement $\vec{d}$.



The net work done on m:

$$W = |\vec{F}||\vec{d}| = m a d \quad , \text{ but } \boxed{v_f^2 = v_i^2 + 2ad} \quad X$$

$$\therefore W_F = m\left(\frac{v_f^2 - v_i^2}{2d}\right)d = \frac{m}{2}\left(v_f^2 - v_i^2\right) \text{ —— } (*)$$

- The quantity $\left[\frac{mv^2}{2}\right]$ is the energy associated with the motion of a moving object: Kinetic energy.

$$K = \frac{mv^2}{2} . \text{ —— Eq (2).}$$

Note that, by definition, K can never be negative. Also note that K, like all forms of energy, is a scalar, not a vector, quantity.

- **Quiz:** A particle of mass m moves with kinetic energy K. If K is tripled, by what factor has the particle's speed increased? $v^* = 3v$ or $\sqrt{3}\,v$?
- Now, Eq (*), page 4, reads:

$$\underbrace{W_F = K_f - K_i = \Delta K}, \text{———— Eq (3).}$$

This equation is known as the Work-Energy theorem:

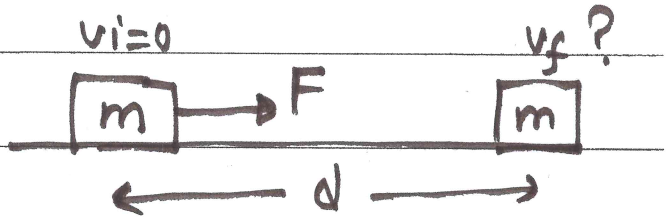
"When the only influence (or change) done by $\vec{F}$ on an object is in its speed, the net work done equals the change in the object's K".

Note that the theorem is limited in it's application, i.e., it is not a general principle. If other changes in the object occur besides its speed, a more general principle must be used. This is the theme of what we are going to discuss thoroughly in ~~page~~ lecture 8.

- Do problem 6.19.

- **Example:** Work-Energy theorem vs Newton's 2^{nd} law:

A 6-kg block initially at rest is pulled to the right along a frictionless, horizontal surface by a constant horizontal force of magnitude 12 N. Find the block's speed after it has moved through a horizontal distance of 3 m.



- Work-Energy approach

$$W_F = Fd = \Delta K = K_f - 0$$

$$\Rightarrow Fd = \frac{m v_f^2}{2}$$

$$\Rightarrow v_f = \sqrt{\frac{2Fd}{m}}$$

- Newton's 2^{nd} law approach

$$\vec{F} = m\vec{a} \Rightarrow \vec{a} = \vec{F}/m$$

$$\therefore |\vec{a}| = F/m \Rightarrow a \text{ is constant}$$

$$\Rightarrow \boxed{v_f^2 = v_i^2 + 2\vec{a}\,\Delta\vec{x}}$$

$$\therefore v_f = \sqrt{\frac{2Fd}{m}}$$

$\Rightarrow$ Substitute the numerical values from the statement

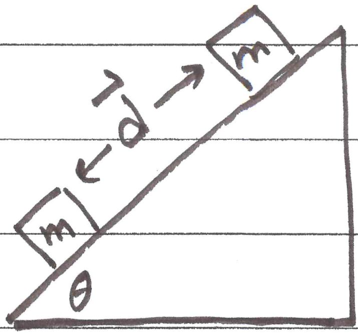
$$v_f = 3.5 \text{ m/s}$$

$\Rightarrow$ Thus, we see that, the work-energy theorem is equivalent to Newton's 2^{nd} law.

- Do problem 6.18: –( )J!

▪ Problem 6.5:

Find the minimum work needed to push $\boxed{m}$ a distance (d) up along the incline (θ).

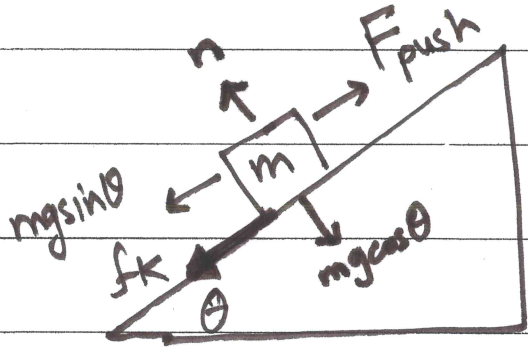


The text has ignored the friction, but we will not do so!

⇒ Notice that the minimum work does occur when $\boxed{m}$ is pushed up at a constant speed $(a=0)$.

$$\therefore F_{push} = mg\sin\theta + f_k$$

$$= mg\sin\theta + \mu_k mg\cos\theta$$



$$W_{F_{push}} = F_{push} * d$$

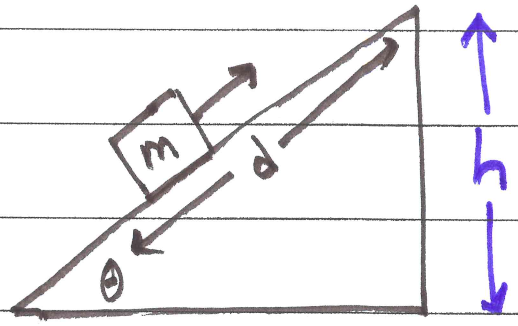
$$W = mg\left[\sin\theta + \mu_k\cos\theta\right] * d \quad \text{—— (i)}$$

▪ If we ignore friction ⇒ $W = mg\left[\sin\theta\right] * d$ —— (ii)

▪ Notice that $W_{(ii)}$ is less than $W_{(i)}$ ✓

▪ Do example 6.2

※ Let's generalize problem 6.5 ※



The work needed to move $\boxed{m}$ up a frictionless ramp of length d at a constant speed is given by eq (ii) page 7:

$$W = mg[d * \sin\theta] = mgh,$$ where h is the height of the ramp.

- The quantity [mgh or mgy] is the energy associated with the vertical height of $\boxed{m}$ relative to the surface of the earth: gravitational potential energy.

$$U = mgy \quad \text{—— Eq (4).}$$

U, like work and K, is a scalar quantity. However, U of $\boxed{m}$ can be positive or negative relative to the surface of the earth, while K can never be negative.

- Using our definition of U in Eq (4), the work-energy theorem can now be rewritten as:

$$\underbrace{W_F = U_f - U_i = \Delta U}$$ — Eq (5).

⇒ To sum up, the influence of the work done on an object appears as a change in the object's Kinetic energy [Eq (3)] or appears as a change in the object's gravitational potential energy [Eq (5)] or both! This will be addressed in lecture 8.

- Example: A mass m is held 3m above the top edge of a well and then dropped into it. The well has a depth of 5m. Choosing the top edge of the well as the $y=0$ point of your coordinate system, what is U of the mass m:

(i) before it is released?

(ii) when it reaches the bottom of the well?

(iii) What is ΔU from release to reaching the bottom of the well?

(i) $U_i = mg\,y_i = +3mg$ J.

(ii) $U_f = mg\,y_f = -5mg$ J.

(iii) $\Delta U = U_f - U_i = -5mg - (+3mg) = -8mg$ J.

(iv) What is ΔK from the release position to the bottom-of-the-well position?

$\Delta K = \frac{m}{2}(v_f^2 - v_i^2)$, $v_f^2 = v_i^2 + 2(-g)(-8)$

$\Rightarrow (v_f^2 - v_i^2) = +16g$

$\therefore \Delta K = +8mg$ J.

Hmm ...! $\Delta U = -8mg$ (Loss!)

$\Delta K = +8mg$ (Gain!)

- Can you get a glimpse of what does this result tell us about?

The sum of the kinetic and potential energies of the object, which is defined as the total mechanical energy $E = K + U$, is conserved:

$$\Delta E = 0 \Rightarrow \boxed{K_i + U_i = K_f + U_f \quad \text{Eq (6)}}$$

Eq (6) is a statement of conservation of mechanical energy.

- Do problem 6.64.

- A 1.5 kg mass has a speed of 20 m/s when it's 15 m above the ground. What is the total energy of the mass?
520.5 J

* Power:

I bet you can now readily calculate the amount of work required to accelerate a 1550-kg car from a standing start to a speed of 26.8 m/s (60 mph). The work is simply ΔK (eq (3)) = 557 kJ.
However, the work requirement is not that interesting to most of us — we'd be more interested in how **quickly** the car is able to reach 60 mph. That is, we'd like to know the **rate** at which the car can do this work.

⇒ **Power** is the rate at which work is done.

$$P_{ave} = \text{average power} = \frac{W}{\Delta t} \qquad \text{Eq (7).}$$

The SI unit of power is the watt (W).
A non-SI power unit is the horsepower (hp):

1 hp = 746 W.

⇒ Returning to the example of the accelerating car, let's assume that the car can reach a speed of 60 mph in 7.1 s. What is the average power needed to accomplish this?

$$P_{ave} = \frac{W}{\Delta t} = \frac{557 \, 10^{+3} \, J}{7.1 \, s} = 78.4 \, 10^{+3} \, W = 105 \, hp.$$

⇒ If you own a car with a mass of 1550 kg that has an engine with 105 hp, you know that it cannot reach 60 mph in possibly 7.1 s. Our calculation is not quite "correct" for several reasons: "efficiency"? and friction - air resistance forces were ignored.

■ Please read example 6.4 and example 6.14.

⇒ For a constant force, $W = |\vec{F}| * |\vec{d}| * \cos\theta$. In this case, $P = \frac{W}{\Delta t} = \frac{|\vec{F}| * |\vec{d}| * \cos\theta}{\Delta t}$

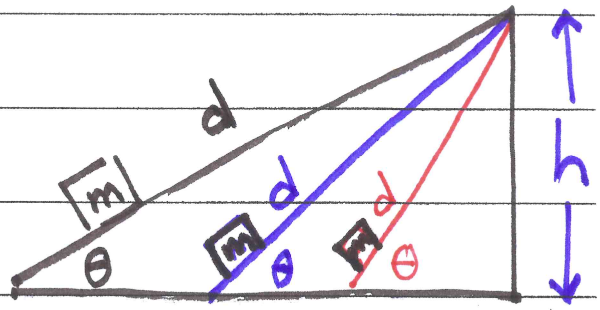
$$P = |\vec{F}||\vec{v}| * \cos\theta \quad \text{——— Eq (8)}$$

where θ is the angle between the force vector and the velocity vector. Therefore, the power is the scalar product of the force vector and the velocity vector.

■ Do problem 6.55.

- Recall problem 6.5 discussed in page 8.

Eq (4) shows that the work done to move m up a ramp is the same regardless of the ramp's length:



$$W = \Delta U = mg\,[d \sin\theta] = mg\,[d \sin\theta] = mg\,[d \sin\theta]$$

$\therefore\ W = mg\,h$ ✓ i.e. the work depends only on the height (not the length) of the ramp.

⇒ Although the work done on the 3 ramps is the same ($W = W = W$), there is something different about the tasks; the time interval during which the work (the job) is done!

⇒ Ramp(d) takes longer to do the work because it's a longer ramp ⇒ $\Delta t > \Delta t > \Delta t$

$$\therefore\ P = \frac{W}{\Delta t} < \frac{W}{\Delta t} < \frac{W}{\Delta t}$$ (numerators are equal)

$\therefore$ The least power is delivered in the longest ramp

- The work done by a "conservative" force on an object moving between any two points is independent of the path taken by the object. The gravitational force is one example of a conservative force. A force that does not **fulfill** this requirement is nonconservative force: friction force.

Problems 6.49 and 6.58 :

(i) 6.49: 2750 W motor lift 385 kg 16 m above: t?

The wording of the question is poor! We need to stress that the task of lifting the object is at a steady speed.

$$\Rightarrow P = \frac{W}{\Delta t} = \frac{\Delta U}{\Delta t} \Rightarrow \Delta t = \frac{\Delta U}{P} = \frac{mgh}{P}.$$

(ii) 6.58: a pump lifts 27 kg of water per minute through 3.5 m. What minimum output (W) must the pump have?

The wording is right! Minimum $\Rightarrow$ no acceleration

$$\Rightarrow P = \frac{W}{\Delta t} = \frac{\Delta U}{\Delta t} = \frac{mgh}{\Delta t} = \left[\frac{m}{\Delta t}\right] gh.$$

Problem 6.56 and our example (page 11):

Car of mass m, accelerates from v_i to v_f in t.

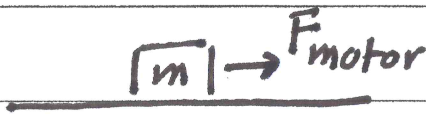
What is the average power delivered by the engine?

(i) $P \equiv$ as eq(7) $= \frac{W}{\Delta t} = \frac{\Delta K}{\Delta t} = \frac{m}{2\Delta t}(v_f^2 - v_i^2)$.

(ii) $P \equiv$ as eq(8) $= \boxed{F \ast V_{ave} = (ma)(V_{ave})}$

$$\times = \boxed{m\left(\frac{v_f - v_i}{\Delta t}\right)\left(\frac{v_f + v_i}{2}\right) = \frac{m}{2\Delta t}(v_f^2 - v_i^2)}$$

▪ Problem 6.60:



The minimum power delivered by the motor to drag $\boxed{m}$ at constant speed v if μ_K is ✓.

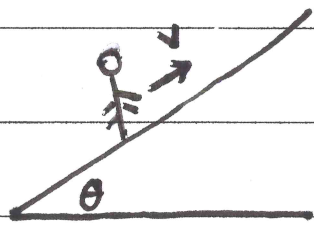
$\Rightarrow \; a = 0 \Rightarrow F_{motor} = f_K = \mu_K mg \Rightarrow W_{motor} = F_{motor} * d$

$\therefore \; P_{motor} = \dfrac{W}{t} = F_{motor} * \dfrac{d}{t} = \mu_K mg\, v$

or $\Rightarrow$ using eq(8) $\Rightarrow P_{motor} = F * v * \cos\theta = F * v = f_K * v$ ✓

▪ Problem 6.68: The stress test for cardiac function!

The patient walks on an inclined treadmill.



The power delivered by the patient:

[m, θ, v] given.

The patient walks at constant $v \Rightarrow F_{*} = mg\sin\theta$

$\therefore \; P_{*} = F_{*} * v_{*} = \underline{(mg\sin\theta)(v)}$.

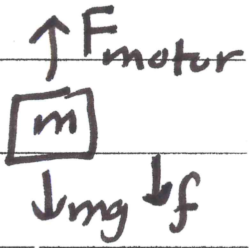
Hmm... to be a good cardiologist, do not ever increase the slope of the treadmill and speed it up while you are checking on a patient with "large inertia"...!

- **Example:** A constant friction force f is retarding the motion of an elevator of mass m.

(i) How much power must a motor deliver to lift the elevator at a constant speed v?

(ii) What power must the motor deliver at the instant the speed of the elevator is v if the motor is designed to provide the elevator with an upward acceleration a?

$\Rightarrow$ (i) $F_{motor} = mg + f$



$\therefore P_{motor} = F * v = (mg + f)v$ —— (1)

$\Rightarrow$ f increases the power necessary to lift the elevator.

$\Rightarrow$ (ii) We expect that more power will be required in this part than in part (i) because the motor must now perform an additional task of accelerating the elevator!

$F_{motor} - f - mg = ma \rightarrow F = m(a+g) + f$

$\therefore P_{motor} = F * v = \left[m(a+g) + f\right]v$ —— (2)

$P_{motor}(2)$ is larger than $P_{motor}(1)$, as expected.