Random Variables

The outcome of a probability experiment is often a count or a measure. When this occurs, the outcome is called a **random variable.**

DEFINITION

A **random variable** x represents a value associated with each outcome of a probability experiment.

The word *random* indicates that x is determined by chance. There are two types of random variables: **discrete** and **continuous.**

DEFINITION

A random variable is **discrete** when it has a finite or countable number of possible outcomes that can be listed.

A random variable is **continuous** when it has an uncountable number of possible outcomes, represented by an interval on a number line.

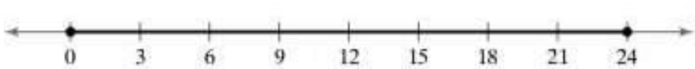
In most applications, discrete random variables represent counted data, while continuous random variables represent measured data. For instance, consider the following example. You conduct a study of the number of calls a telemarketing firm makes in one day. The possible values of the random variable x are 0, 1, 2, 3, 4, and so on. Because the set of possible outcomes $\{0, 1, 2, 3, \ldots\}$ can be listed, x is a discrete random variable. You can represent its values as points on a number line.

Number of Calls (Discrete)

x can be any whole number: 0, 1, 2, 3, . . .

A different way to conduct the study would be to measure the time (in hours) the telemarketing firm spends making calls in one day. Because the time spent making calls can be any number from 0 to 24 (including fractions and decimals), x is a continuous random variable. You can represent its values with an interval on a number line.

Hours Spent on Calls (Continuous)



x can be any value between 0 and 24.

When a random variable is discrete, you can list the possible values the variable can assume. However, it is impossible to list all values for a continuous random variable.

EXAMPLE 1

Discrete Variables and Continuous Variables

Determine whether each random variable x is discrete or continuous. Explain your reasoning.

1. Let x represent the number of Fortune 500 companies that lost money in the previous year.
2. Let x represent the volume of gasoline in a 21-gallon tank.

SOLUTION

1. The number of companies that lost money in the previous year can be counted. The set of possible outcomes is

$$\{0, 1, 2, 3, \ldots, 500\}.$$

So, x is a *discrete* random variable.

2. The amount of gasoline in the tank can be any volume between 0 gallons and 21 gallons. So, x is a *continuous* random variable.

TRY IT YOURSELF 1

Determine whether each random variable x is discrete or continuous. Explain your reasoning.

1. Let x represent the speed of a rocket.
2. Let x represent the number of calves born on a farm in one year.
3. Let x represent the number of days of rain for the next three days (see page 211).

Answer: Page A34

It is important that you can distinguish between discrete and continuous random variables because different statistical techniques are used to analyze each. The remainder of this chapter focuses on discrete random variables and their probability distributions. Your study of continuous probability distributions will begin in Chapter 5.

Discrete Probability Distributions

Each value of a discrete random variable can be assigned a probability. By listing each value of the random variable with its corresponding probability, you are forming a **discrete probability distribution.**

DEFINITION

A **discrete probability distribution** lists each possible value the random variable can assume, together with its probability. A discrete probability distribution must satisfy these conditions.

In Words	In Symbols
1. The probability of each value of the discrete random variable is between 0 and 1, inclusive.	$0 \le P(x) \le 1$
2. The sum of all the probabilities is 1.	$\Sigma P(x) = 1$

4.2

Random Variable (r.v.)

Chapter 4
lecture "15"

المتغيرات العشوائية المنفصلة والمتصلة.

اقترانات

A function from the sample space to a set of real numbers. We denote the r.v. by:

X or Y or T or ...

تستخدم هذه الرموز وغيرها → capital

e.g.) When tossing a fair coin 2 times, define the r.v. X to be the no. of heads obtained.

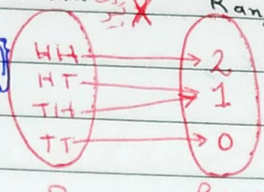
يدل على عدد h

Sol: $\Omega = \{HH, HT, TT, TH\}$

P(X=0)

$\{(0, \frac{1}{4}), (1, \frac{1}{2}), (2, \frac{1}{4})\}$

التوزيع الاحتمالي



Ω Sample space — يربط بأعداد
Range (support)

function في الدالة — صورة العنصر HH هي

$X(HH) = 2$

$X(HT) = X(TH) = 1$

$X(TT) = 0$

$x \in \{0, 1, 2\}$

Discrete مجموعة معدودة

Note: If the support is a countable set $\{x_1, x_2, ...\}$, then the r.v. is called discrete r.v. (قابلة للعد) سندرسها بالتفصيل

If the support is an interval $a < x < b$, then the r.v. is called Continuous r.v. (قابلة للقياس وليس العد) ولن ندرس حالات منها

(مغلقة او مفتوحة) فترة او اتحاد فترات (intervals)

مثل حجم البنزين

* Discrete r.v.:

* Prob. distribution:

التوزيع الاحتمالي — مفتاح الحل

x	x_1	x_2	x_3	...
$P(X=x)$	$P(X=x_1)$	$P(X=x_2)$	$P(X=x_3)$	...

يشبه الى حد ما $r.f = \frac{f}{\sum f}$

$P(X=x) \geq 0$ موجبة

$0 \leq P(X=x) \leq 1$

$\sum_x P(X=x) = 1$ مجموع

* Find the prob. distribution in the previous example.

Sol)

x	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

يعني ما احتمال ظهور 2 heads الجواب $\frac{1}{4}$

احتمال TT — احتمال HT او TH — احتمال HH

COLLEGE

تعريف

Def): $f(x) = P(X=x)$ is called a prob. density function (p.d.f) if:

a) $f(x) = P(X=x) \ge 0 \quad \forall x$

for all تُقرأ

b) $\sum_x P(X=x) = 1$

لازم يحقق الشرطين معاً.

eg) If $P(X=x) = k * x$,

$x = 1, 2, 3, 4$

is a p.d.f find:

i) the value of k.

ii) $P(X=3)$

iii) $P(1 < X < 4)$

iv) $P(X = 2.5)$

v) $P(X = 5)$

vi) $P(X > 1 \mid X < 4)$

Sol)

x	1	2	3	4
$P(X=x)$	k	$2k$	$3k$	$4k$

i) $k + 2k + 3k + 4k = 1$

$10k = 1$

$\boxed{k = 0.1}$

x	1	2	3	4
$P(X=x)$	0.1	0.2	0.3	0.4

ii) $P(X=3) = 0.3$

iii) $P(1 < X < 4) = P(X=2) + P(X=3)$

$= 0.2 + 0.3 = 0.5$

iv) $P(X = 2.5) = 0$

v) $P(X = 5) = 0$

vi) $P(X > 1 \mid X < 4)$

$= \dfrac{P(X > 1 \cap X < 4)}{P(X < 4)}$

1   4

$\dfrac{5}{6} = \dfrac{0.5}{0.6} = \dfrac{P(1 < X < 4)}{P(X < 4)}$

If $P(X=x) = k \cdot \left(\frac{2}{3}\right)^x$, $x = 0, 1, 2, \ldots$

is a p.d.f , then find

i) The value of k ii) $P(X \leq 9)$:

Sol)

x	0	1	2	3	...
$P(X=x)$	k	$k\left(\frac{2}{3}\right)$	$k\left(\frac{2}{3}\right)^2$	$k\left(\frac{2}{3}\right)^3$	...

تتبعي i)

التسلسل ، $k + k\left(\frac{2}{3}\right) + k\left(\frac{2}{3}\right)^2 + k\left(\frac{2}{3}\right)^3 + \cdots = 1$

الهندسي (G.P)

* Geometric Series (G.P)

$$a + ar + ar^2 + ar^3 + \cdots$$

$\boxed{1}$ $S_n = \frac{a(1-r^n)}{1-r}$ $\boxed{2}$ $S_\infty = \frac{a}{1-r}$; $|r| < 1$

$a = k$ / $r = \frac{2}{3}$ $\Rightarrow \therefore S_\infty = 1$

$\rightarrow S_\infty = \frac{a}{1-r} = \frac{k}{1-\frac{2}{3}} = 1 \rightarrow \frac{k}{1/3} = 1 \rightarrow k = \frac{1}{3}$

$P(X=x) = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^x$, $x = 0, 1, 2, \ldots$

ii) $P(X \leq 9) = P(X=0) + P(X=1) + \cdots + P(X=9)$

$\underbrace{k + k\left(\frac{2}{3}\right) + k\left(\frac{2}{3}\right)^2 + \cdots + k\left(\frac{2}{3}\right)^9}_{n=10}$

(G.P) $\rightarrow S_n = \frac{a(1-r^n)}{(1-r)} \Rightarrow S_{10} = \frac{\frac{1}{3}\left(1 - \frac{2}{3}^{10}\right)}{\left(1 - \frac{2}{3}\right)}$

eg) For the following Prob. distribution:

x	1	2	3	4
$P(X=x)$	$2a$	0.3	a	0.1

Find a : sol) $2a + 0.3 + a + 0.1 = 1$

$3a + 0.4 = 1$

$3a = 0.6 \rightarrow \boxed{a = 0.2}$

eg) when throwing a fair die two times, define the r.v. S to be the sum of the 2 numbers obtained. Find the prob. distribution of S.

sol $\Omega = \{(1,1), (1,2), \ldots, (6,6)\}$, $n\Omega = 36$

S	2	3	4	5	6	$\boxed{7}$	8	9	10	11	12
$P(S=s)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

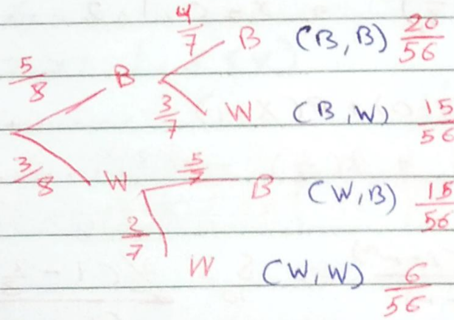
symetric distribution

eg)

5	3
B	W

2 balls are selected at random let the r.v Y be the no. of black balls obtained Find the prob. distribution of Y

لانه سحبنا اكثر من مرة



Y	0	1	2
$P(Y=y)$	$\frac{6}{56}$	$\frac{30}{56}$	$\frac{20}{56}$

mean $= \frac{\sum fx}{\sum f} \rightarrow 1$

* The expected value or the mean

Lecture "16"

* $\mu = E(X) = \sum_x x \cdot P(X=x)$ → نضرب كل مشاهدة باحتمالها ونجمعهم

→ related to the population

eg. Find the μ for:

x	1	2	3	4
$P(X=x)$	0.4	0.3	0.2	0.1

(2.5 between 2 and 3; اكثر under 0.4, 0.3; أقل under 0.2, 0.1)

Sol) $\mu = 1(0.4) + 2(0.3) + 3(0.2) + 4(0.1)$

$0.4 + 0.6 + 0.6 + 0.4 = 2$

لاحظ : لما كانت raw data كانتا 1, 2, 3, 4 واردنا نجد mean
يكون بالمنتصف فيه (2.5) لكن الآن لم يعد 2.5 داعا لوجود
للأكثر احتمالية فيزيح باتجاه الاكبر P(X=x) فيكون 2 ولكن يبعد ~ 2.5
اذا كان ~ الـ distribution symmetric ⟸ معلومة مهمة جداً بالامتحان

eg)

x	1	2*	3
$P(X=x)$	a	b	a

symetric

Find $\mu = E(X)$

sol)) $\therefore \mu = 2$

eg.

x	1	2	3	4
$P(X=x)$	a	0.3	b	0.1

if $E(X) = 2$ Find a & b

sol) $\mu = 1(a) + 2(0.3) + 3(b) + 4(0.1) = 2$

* $2 = a + 0.6 + 3b + 0.4$

$\therefore a + 3b = 1$ ---- ①

but $\sum_x P(X=x) = 1$

* $a + 0.3 + b + 0.1 = 1 \Rightarrow a + b + 0.4 = 1$

$a + b = 0.6$ ---- ②

① − ② $\Rightarrow 2b = 0.4 \rightarrow \boxed{b = 0.2} \rightarrow$ نعوض في ② $\rightarrow a + 0.2 = 0.6$

$\rightarrow \boxed{a = 0.4}$

* Properties of $E(X)$:

$E(2X) = 2E(X)$

$2X = 2, 4, 6, 8, 10$

$X = 1, 2, 3, 4, 5$

$E(X) = \text{mean} = 3$

$Y: 2, 2, 2, 2, 2$

mean $= 2$

$E(Y) = 2$

① $E(a) = a$

الثابت

② $E(aX) = a\,E(X)$

sum or difference

③ $E(X \mp Y) = E(X) \mp E(Y)$ $(\mu_{X+Y} = \mu_X + \mu_Y)$

Independent and Dependent Random Variables

④ $E(g(x)) = \sum_x g(x) \cdot P(X=x)$

للتوضيح الخاصية الرابعة.

eg.

X	1	2	3	4
$P(X=x)$	0.4	0.3	0.2	0.1

Find i) $E(X) = 1(0.4) + 2(0.3) + 3(0.2) + 4(0.1)$

$= 0.4 + 0.6 + 0.6 + 0.4 = 2$

ii) $E(X^2) = 1^2(0.4) + 2^2(0.3) + 3^2(0.2) + 4^2(0.1)$

متغير ال function

$= 1(0.4) + 4(0.3) + 9(0.2) + 16(0.1)$

$=$

iii) $E\left(\frac{1}{X}\right) = \frac{1}{1}(0.4) + \frac{1}{2}(0.3) + \frac{1}{3}(0.2) + \frac{1}{4}(0.1)$

iv) $E(e^X) = e^1(0.4) + e^2(0.3) + e^3(0.2) + e^4(0.1)$

للتوضيح أول ثلاثة خصائص

* eg) If $\mu = E(X) = 10$ find:

i) $E(\mu) = E(10) = 10$

ii) $E(E(X)) = E(10) = 10$

iii) $E(2X+3) = E(2X) + E(3) = 2E(X) + 3 = 20 + 3 = 23$

iv) $E\left(1 - \frac{X}{2}\right) = E(1) - E\left(\frac{X}{2}\right) = 1 - \frac{1}{2}E(X) = 1 - \frac{1}{2}(10) = 1 - 5 = -4$

* The variance , σ^2

طريقة طويلة $\sigma^2 = Var(X) = E(X-\mu)^2 \Rightarrow$ اعتمدها الكتاب

or
أسهل $Var(X) = E(X^2) - (E(X))^2$

$S.D(X) = \sigma = \sqrt{Var(X)}$

Note: $E(X - E(X))^2 = E(X^2) - (E(X))^2 = Var(X)$

eg) Find Var(X) & S.D(X) for

X	1	2	3	4
$P(X=x)$	0.4	0.3	0.2	0.1

* $E(X) = 0.4 + 0.6 + 0.6 + 0.4 = 2$

* $E(X^2) = 0.4 + 1.2 + 1.8 + 1.6 = 5$

sol) $Var(X) = E(X^2) - (E(X))^2 = 5 - (2)^2 = 5 - 4 = 1$

$S.D(X) = \sqrt{Var(X)} = \sqrt{1} = 1$

* properties of Var(X) :

(1) $Var(a) = 0$

(2) $Var(aX) = a^2\, Var(X)$

(3) $Var(aX \mp b) = Var(aX)$ ما بتأثر بالجمع أو الطرح

(4) $E(X^2) = Var(X) + (E(X))^2 = \sigma^2 + \mu^2$

مشتقة من
$Var(X) = E(X^2) - \cancel{(E(X))^2}$
$+(E(X))^2 \qquad \cancel{+(E(X))^2}$
$\Rightarrow E(X^2) = Var(X) + (E(X))^2$

(5) $Var(X \mp Y) = Var(X) + Var(Y)$; if X & Y are $\boxed{\text{indipendent}}$

when the value of X doesn't affect the value of Y , بنقول انهم ما بأثروا ببعض

eg) If $\mu = 10$, $\sigma^2 = 3$, then find:

i) $E(Var(X)) = E(3) = 3$

ii) $Var(E(X)) = Var(10) = 0$

iii) $Var(Var(X))$ $Var(3) = 0$

iv) $Var(2X - 3) = Var(2X) = 4\,Var(X) = 4 * 3 = 12$

v) $S.D(3 - 2X)$ $\sqrt{Var(-2X)} = \sqrt{Var(X) * 4} = \sqrt{12}$

vi) $E(X - \overset{\mu}{10})^2 = \boxed{\sigma^2} = (E(X) - 10)^2 + Var(X - 10)$

$= (10 - 10)^2 + Var(X) = 0 + 3 = 3$

لاحظ أن $E(X - \mu)^2 = \sigma^2$.

vii) $E(X^2) = \mu^2 + \sigma^2 = (10)^2 + 3 = \underline{103}$

viii) $E(X-2)^2 = (E(X-2))^2 + Var(X-2)$

$= (E(X) - 2)^2 + Var(X) = (10-2)^2 + 3 = \boxed{67}$

او بطريقة اخرى

$E(X^2 - 4X + 4) = E(X^2) - 4E(X) + 4 = 103 - 4(10) + 4$

$103 - 40 + 4$

$\boxed{67}$

* Special Distributions
- Discrete
  - Binomial
  - Poisson
- Continuous
  - Normal ← مهم كتير
  - t - distribution
  - chi-square distribution

Binomial Experiments

There are many probability experiments for which the results of each trial can be reduced to two outcomes: success and failure. For instance, when a basketball player attempts a free throw, he or she either makes the basket or does not. Probability experiments such as these are called **binomial experiments.**

DEFINITION

A **binomial experiment** is a probability experiment that satisfies these conditions.

1. The experiment has a fixed number of trials, where each trial is independent of the other trials.
2. There are only two possible outcomes of interest for each trial. Each outcome can be classified as a success (S) or as a failure (F).
3. The probability of a success is the same for each trial.
4. The random variable x counts the number of successful trials.

Notation for Binomial Experiments

Symbol	Description
n	The number of trials
p	The probability of success in a single trial
q	The probability of failure in a single trial $(q = 1 - p)$
x	The random variable represents a count of the number of successes in n trials: $x = 0, 1, 2, 3, \ldots, n$.

In a binomial experiment, success does not imply something good occurred. For instance, in an experiment a survey asks 1012 people about identity theft. A success is a person who was a victim of identity theft.

Here is an example of a binomial experiment. From a standard deck of cards, you pick a card, note whether it is a club or not, and replace the card. You repeat the experiment five times, so $n = 5$. The outcomes of each trial can be classified in two categories: S = selecting a club and F = selecting another suit. The probabilities of success and failure are

$$p = \frac{1}{4} \quad \text{and} \quad q = 1 - \frac{1}{4} = \frac{3}{4}.$$

The random variable x represents the number of clubs selected in the five trials. So, the possible values of the random variable are $x = 0, 1, 2, 3, 4, 5$. For instance, if $x = 2$, then exactly two of the five cards are clubs and the other three are not clubs. An example of an experiment with $x = 2$ is shown at the left. Note that x is a discrete random variable because its possible values can be counted.

4.2

The Binomial distribution

توزيع ذي الحدين

Lecture '17'

female , Male او H , T مثلاً : [2 outcomes] لما يكون المخرجات

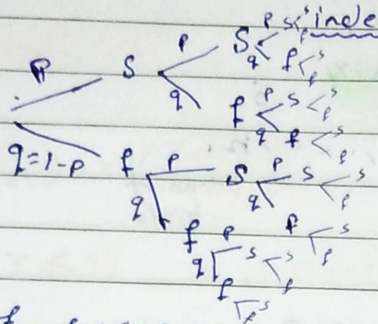
1 = مجموع احتمالاتهم fail او success (تجربة برنولي)

والنتيجتين ما بأثر ببعض لازم يكونوا independent

ما بأثروا على بعض .

* لو اكررنا 3 مرات لازم نلاقي

حل بدل Tree diagram



$p + q = 1$

$q = 1 - p$

Throwing a fair die 1 time

X : no. of 5

$p = \frac{1}{6} \to 5$

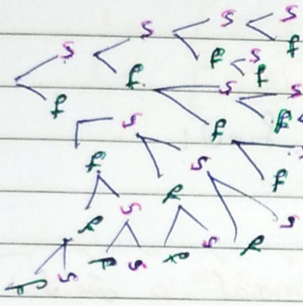
$q = \frac{5}{6} \to$ not 5

$* p = P(X=1)$

$* q = P(X=0)$

* X : no of successes

* n : no of trials & here n = 4



$P(X=4) = p^4$ → no. of success

$P(X=3) = p^3 \cdot q^1$ → no. of fails

$P(X=3) = p^3 \cdot q^1$

$P(X=3) = p^3 \cdot q^1$

$P(X=3) = p^3 \cdot q^1$

$\therefore P(X=3) = 4\, p^3 \cdot q^1$

p ل القوة number of success

q ل القوة no. of fails

X: no. of success

$$P(X=3) = \binom{4}{3} p^3 \cdot q^{4-3}$$

لأن بدنا اختار 3 من اصل 4 رميات

رح نصير نستغني بدل ال Tree

السؤال الأهم : كيف اعرف انه Binomial

اذا رح أكرر التجربة اكثر من 3 مرات وفي كل مرة بجرب 2 outcomes العكس

عكس بعض نجاح ورسوب .. بنت وولد .. اطول من 160 اقل من 160

رميا حجر نرد طلع even ولا odd وهكذا يكونوا independent ما بأثروا على بعض

لازم تتحققوا الثلاث شروط حتى أحكي عنه Binomial .

If we have n-independent trials $(n \geq 3)$ & the outcomes in each trial are success (S) or fail (F) only .

Let X be the no. of successes, then we say that X follows a binomial distribution is denoted by $X \sim Bin(n, p)$;

Where n : number of trials. p : prob of successful in each trial.

بالتجربة الواحدة

* كل محاولة للصياد تحتمل احتمالين ، فإما ان ينجح في إصابة الهدف ، وإما ان يفشل في ذلك . وهذه المحاولة تسمى تجربة برنولي وفضاؤها العيني مكون من نتيجتين منفصلتين ، اما نجاح واما فشل .

وفي حال تكررت عدة مرات (n) تكون مستقلة ومتماثلة (نتيجة احدى التجارب لا تؤثر في سواها) ،

وكان احتمال النجاح ثابتاً في كل مرة (P) فيسمى هذا النوع من التجارب .. Binomial

e.g.) Thus, if $X \sim Bin(n, p)$, then
The p.d.f is given by:

$$P(X=x) = \binom{n}{x} p^{x} \cdot q^{n-x} \quad ;$$

$q = 1-p$ } $x = 0, 1, 2, \ldots, n.$

Non Negative Integers

no. of trials

Notes: ① $\mu \sim E(X) = np$
② $\sigma^2 = Var(X) = n.p.q$
③ $\sigma = \sqrt{Var(X)} = \sqrt{n.p.q}$

e.g.) When tossing a fair coin 10 times find:
a) the prob. of getting:
i) exactly 8 heads
ii) at least 9 heads.
iii) at least 2 heads
iv) at most 1 head
v) at most 8 heads
vi) at most 2 heads given that at least 1 head
b) The expected no. of heads, the varience & the standard deviation.

اول شي هاد Binomial لأنه
① $n \geq 3$; $n = 10$
② رمي قطعة النقد ١٠ مرات يطلع اول مرة h ما إلها تأثير بالتانية
③ لما رمي يطلع بس حالتين h / t

sol) $n = 10$, $P = 0.5$ → احتمال انه يطلع head بحسبة الواحدة
X: no. of heads
$X \sim Bin(10, 0.5)$

a) i) $P(X=8) = \binom{10}{8}(0.5)^{8}(1-0.5)^{2} = \frac{10!}{8!\,2!}\left(\frac{1}{2}\right)^{8}\left(\frac{1}{2}\right)^{2} = \frac{45}{1024}$

ii) $P(X \geq 9) = P(X=9) + P(X=10)$
$\binom{10}{9}(0.5)^{9}(1-0.5)^{1} + \binom{10}{10}(0.5)^{10}(1-0.5)^{0}$

$= \frac{11}{1024}$

iii) $P(X \geq 2) = 1 - P(X \leq 2) = 1 - P(X \leq 1)$

$1 - (\cancel{P(X=2)} + P(X=1) + P(X=0))$

$1 - \left(\cancel{\binom{10}{2}(0.5)^2(0.5)^8} + \binom{10}{1}(0.5)^1(0.5)^9 + \binom{10}{0}(0.5)^0(0.5)^{10}\right)$

$1 - \left(\frac{11}{1024}\right) = \frac{1013}{1024}$

iv) $P(X \leq 1) = P(X=1) + P(X=0) = \frac{11}{1024}$

v) $P(X \leq 8) = 1 - P(X > 8) = 1 - P(X \geq 9)$

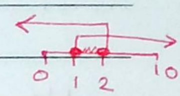
$1 - [P(X=9) + P(X=10)]$

$1 - \left[\binom{10}{9}(0.5)^9(0.5)^1 + \binom{10}{10}(0.5)^{10}\cancel{(0.5)^0}\right]$

$1 - \left(\frac{11}{1024}\right) = \frac{1013}{1024}$

vi) $P(X \leq 2 \mid X \geq 1) = \frac{P(X \leq 2) \cap P(X \geq 1)}{P(X \geq 1)}$

$= \frac{P(1 \leq X \leq 2)}{P(X \geq 1)} = \frac{P(X=1) + P(X=2)}{1 - P(X<1)}$



$\frac{P(X=1) + P(X=2)}{1 - P(X=0)} = \frac{\binom{10}{1}(0.5)^1(0.5)^9 + \binom{10}{2}(0.5)^2(0.5)^8}{1 - \left(\binom{10}{0}(0.5)^0(0.5)^{10}\right)}$

b) $E(X) = np = (10)(0.5) = 5$

$var(X) = npq = 10(0.5)(1-0.5) = 2.5$

$S.D = \sqrt{var(x)} = \sqrt{2.5} \approx$

Note * $P(X<2) + P(X \geq 2) = 1$

$\rightarrow P(X<2) = 1 - P(X \geq 2)$

* If $X \sim Bin(n, p)$; n = no. of trials
p : prob. of successful

Then:

① $P(X = x) = \binom{n}{x} \cdot p^x \cdot q^{n-x}$, $x = 0, 1, 2, \ldots n$
, $q = 1 - p$

لا يمكن ان تكون سالب   لا تتجاوز (n)

② $\mu = E(X) = n \cdot p$

③ $\sigma^2 = Var(X) = n \cdot p \cdot q$ ④ $\sigma = \sqrt{Var(X)} = \sqrt{npq}$

eg) If $X \sim Bin(3, p)$ & $P(X \geq 1) = \frac{19}{27}$, find $Var(X)$

لازم نوجد p اولاً حسبنا $q = 1 - p$

sol) $x = 0, 1, 2, 3$

$P(X \geq 1) = 1 - P(X < 1) = 1 - (P(X = 0))$

$\Rightarrow \cancel{1} - \left(\binom{3}{0} p^0 (1-p)^3\right) = \frac{19}{27} - 1$

$\frac{8}{27} = 1 (q)^3 \longrightarrow q = \frac{2}{3} \rightarrow p = 1 - \frac{2}{3} = \frac{1}{3}$

$Var(X) = npq = 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{2}{3}$

eg) If $X \sim Bin(n, p)$, $\mu = 2$ & $\sigma^2 = 1.6$ Find n & p

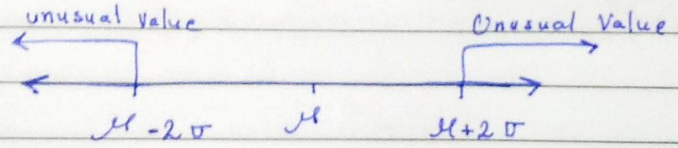
sol) $\mu = E(X) = \boxed{2 = np}$

$\sigma^2 = npq = 2q = 1.6 \rightarrow q = 0.8$

$p = 1 - q = 1 - 0.8 = \boxed{0.2 = p}$

subsitution $\rightarrow$ $np = 2 \rightarrow 0.2n = 2 \rightarrow \boxed{n = 10}$

EXAMBLE ← Page 232 ارجع الك



Using the tables

* هذا الجدول تم إلغاء الاعتماد عليه

$P(X \le k)$ from the tables.

n: ...

x	P: 0.1	0.2	...
0			
1			
2			
3			
⋮			
n			

((غير موجود الا في الجدول)) Page 670

eg) If $X \sim Bin(10, 0.4)$ find:

i) $P(X \le 6)$ من الجدول $\Rightarrow 0.945$

مثلاً $X \sim Bin(5, 0.35)$ find $P(X=3)$ مباشرة من الجدول. أما لإيجاد $P(X \le 3)$ $= P(X=0) + P(X=1) + \dots P(X=3)$

ii) $P(X < 7) = P(X \le 6) = 0.945$

iii) $P(X > 6) = 1 - P(X \le 6) = 1 - 0.945 = 0.055$

iv) $P(X \ge 7) = 1 - P(X < 7) = 1 - P(X \le 6) = 1 - 0.945 = 0.055$

v) $P(3 \le X \le 8) = P(X \le 8) - P(X \le 2) = \dots$

$x: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ ($0 = 0$, $10 = n$)

vi) $P(3 \le X < 8) = P(X \le 7) - P(X \le 2) = \dots$

vii) $P(3 < X < 8) = P(X \le 7) - P(X \le 3) = \dots$

$x: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

viii) $P(3 < X \le 8) = P(X \le 8) - P(X \le 3) = \dots$ من الجدول

$x: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

ix) $P(X \ge 3 \mid X \le 8) = \dfrac{P(X \ge 3) \cap P(X \le 8)}{P(X \le 8)} = \dfrac{P(3 \le X \le 8)}{P(X \le 8)}$

$x: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ $\quad \dfrac{P(X \le 8) - P(X \le 2)}{P(X \le 8)}$

x) $P(X = 6) = P(X \le 6) - P(X \le 5) = \dots$

* eg) A family has 5 children. What is the prob. that 3 children are females.

X: no. of females.

$X \sim Bin(5, \frac{1}{2})$

$P(X=3) = P(X \le 3) - P(X \le 2)$ من الجدول

$=$

EXAMPLE 4 ← (Page 228) بالرجوع الى الكتاب

Finding a Binomial Probability

A survey...

Randomly select 100 adults ⇒ $n = 100$, $P = 0.26$, $q = 0.74$

$P(X = 35) = \binom{100}{35} (0.26)^{35} (0.74)^{100-35} = 0.0115763$

لاحظ انها اقل من 0.05 ← $\simeq 0.012$

This can be considered an unusual event

EXAMPLE 5 ← Page 229 وما تنسوا الرجوع للكتاب

* في الكتاب الجديد ما اعتمد جداول c.d.f لايجاد مثلاً $P(X \leq 9)$ مباشرة من الجدول.
وانما اعتمد جداول p.d.f يعني بس تطلع $P(X=9)$ او $P(X=7)$ او $P(X=2)$
اذا بدك أقل من 9 لازم توخذ من 0 لغاية ال 9 وتجمعهم مع بعض في Table
الكتاب اعتمد جداول p.d.f كل n الها جدول لغاية 25 يعني لازم نرجع للجداول
ونصير نطلع منها .prob بحيث مثلاً في الجدول

n	x	0.1	0.5	0.10	....
2	0			⋮	
	1	...	...	0.180	
	2				
3	0				
	1				
⋮	2				
⋮	⋮				

هذا العدد يمثل الاحتمالية

عندما $n = 2$ ، $P = 0.30$

$\therefore P(X = 2) = 0.810$

تقاطعهم بعطي هاد الرقم.

EXAMPLE 7 ← (Page 231) بالرجوع الى الكتاب

$P = 0.62$: $x \geq 65$ $\quad n = 6$, $P = 0.62$, $q = 0.38$

$q = 0.38$: $x < 65$

x	0	1	....	6
$P(X=x)$	$\binom{6}{0}(0.62)^0(0.38)^6$	$\binom{6}{1}(0.62)^1(0.38)^5$	....	...

x	0	1	2	3	4	5	6
$P(X=x)$	0.003	0.029	0.120	0.262	0.320	0.209	0.057

Now, you can graph the prob. distribution using a histogram..

بالرجوع الى الكتاب من Page 670 ... أوجد

$x \sim Bin(5, 0.35)$ find i) $P(X=3) = 0.181$

ii) $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$ من الجدول.

iii) $P(X > 3) = P(X=4) + P(X=5)$

vi) $P(x<2 \mid x>3) = \dfrac{P(\phi)}{P(x>3)} = 0$

vii) $P(x=2.5) = 0$ ← لانه غير موجود في الجدول

* prob. distribution function (p.d.f)

→ * The cumulative distribution function (c.d.f):-

$F(a) = P(X \leq a)$

eg)

x	1	2	3	4
$P(X=x)$	0.4	0.3	0.2	0.1

Find:-

i) $F(1) = P(X\leq 1) = 0.4$

ii) $F(2) = P(X\leq 2) = P(x=1)+P(x=2) = 0.4+0.3 = 0.7$

iii) $F(3) = P(X\leq 3) = 0.4+0.3+0.2 = 0.9$

iv) $F(4) = P(X\leq 4) = 0.4+0.3+0.2+0.1 = 1$

اي رقم يتعدى الارقام الموجوده بالجدول يعني صوره الي اكبر من 4

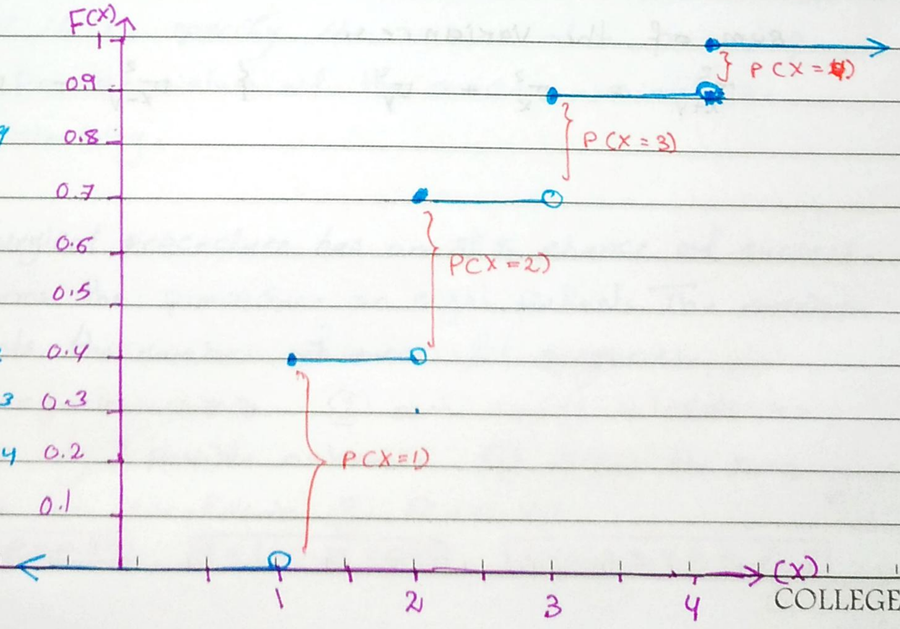
v) $F(5) = P(X\leq 5) = 1$ يكون ال prob. c.d.f يساوي واحد

vi) $F(0) = P(X\leq 0) = 0$ Zero مايوجد اقل من 0 لذلك يكون

vii) $F(2.1) = P(X\leq 2.1) = P(X\leq 2) = 0.7$

viii) $F(2.5) = P(X\leq 2.5) = P(X\leq 2) = 0.7$

ix) $F(2.9) = P(X\leq 2.9) = P(X\leq 2) = 0.7$



* non-decresing

اما متزايد أو ثابت

$$F(x) = \begin{cases} 0, & x<1 \\ 0.4, & 1\leq x<2 \\ 0.7, & 2\leq x<3 \\ 0.9, & 3\leq x<4 \\ 1, & x\geq 4 \end{cases}$$

Finding an Expected Value

At a raffle 1500 tickets are sold a $2 each for four prizes of $500, $250, $150, and $75. You buy one ticket, find the expected value and interpet its meaning.

Solution))

1	1	1	1	1496
$\boxed{500}$	$\boxed{250}$	$\boxed{150}$	$\boxed{75}$	$\boxed{0}$
يربح 500-2	رح يربح 250-2	رح يربح 150-2	رح يربح 75-2	رح يخسر 2

Gain, X	498	248	148	73	-2
P(X=x)	$\frac{1}{1500}$	$\frac{1}{1500}$	$\frac{1}{1500}$	$\frac{1}{1500}$	$\frac{1496}{1500}$

$$E(x) = \sum_x x P(x) = 498\left(\frac{1}{1500}\right) + 248\left(\frac{1}{1500}\right) + 148\left(\frac{1}{1500}\right) + 73\left(\frac{1}{1500}\right) + -2\left(\frac{1496}{1500}\right) = -\$1.35$$

Negative

you can expect to lose an average of $1.35 for each ticket you buy.

Note that the variance of the difference is the sum of the variances.

$$\sigma^2_{X+Y} = \sigma^2_x + \sigma^2_y \quad \{ \quad \sigma^2_{X-Y} = \sigma^2_x + \sigma^2_y$$

Binomial distribution : $X \sim Bin(n,p)$

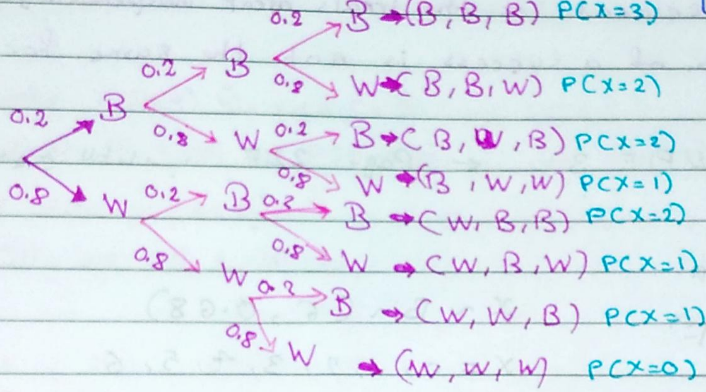
n: no. of trials , p: prob. of success in each trial

Example 3 balls are selected at ranom with replacement, Let X be the no. of black balls obtained.

Note that with replacement mean [independent]

B	W
2	8

sol)



X: no. of B

لاحظ أن :

$p = 0.2$

$q = 0.8$

* $P(X=3) = (0.2)^3$ * $P(X=2) = 3(0.2)^2 \cdot (0.8)^1$
* $P(X=1) = 3(0.2)^1 \cdot (0.8)^2$ * $P(X=0) = (0.8)^3$

$P(X=2) = \binom{3}{2}(0.2)^2 \cdot (0.8)^{3-2}$ بالتعويض على القانون لاحظت

$P(X=x) = \binom{n}{x}(p)^x \cdot (1-p)^{n-x}$ ، القانون :

Example. Determine whether each experiment is a binomial experiment. If it is, specify the values of n, p, and q, and list the possible values of the random variable x. If it isn't, explain why.

1. A certain surgical procedure has an 85% chance of success, A doctor performs the procedure on eight patients. The random variable represents the number of successful surgeries.

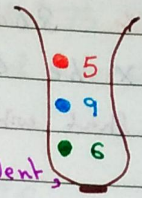
sol) (1) eight surgeries $n \geq 3$ (2) each surgery is independent (3) There are only 2 possible outcomes (4) either the surgery is a success or it is failure (5) $P = 0.85$

$\boxed{n=8}$ $\boxed{P=0.85}$ $\boxed{q = 1-0.85 = 0.15}$ $\boxed{x = 0,1,2,3,4,5,6,7,8}$

[2] A jar contains five red marbles, nine blue marbles, and six green marbles. You randomly select three marbles from the jar, without replacement. The random variable represents the number of red marbles

X: no. of R

$< \begin{matrix} R \\ R' \end{matrix}$



without replacement = the trials aren't independent, and the prob. of a success is not the same for each trial.

EXAMPLE 3 ← Page 227 * عليهِ الرجوع للكتاب

$\underline{\underline{n = 6}}$

$p = 0.68$ F

$q = 0.32$ F'

$X \sim Bin(6, 0.68)$

$X = 0, 1, 2, 3, 4, 5, 6$

Construct a binomial probability distribution ... who respond $\underline{\underline{yes}}$ → F ✓

X	0	1	2	3	4	5	6
P(X=x)	0.001	0.014	0.073	0.206	0.328	0.279	0.099

* $P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$, $q = 1-p$
* $P(X=0) = \binom{6}{0}(0.68)^0 (0.32)^6$
* $P(X=1) = \binom{6}{1}(0.68)^1 (0.32)^5$
* $P(X=2) = \binom{6}{2}(0.68)^2 (0.32)^4$
* $P(X=3) = \binom{6}{3}(0.68)^3 (0.32)^3$
* $P(X=4) = \binom{6}{4}(0.68)^4 (0.32)^2$
* $P(X=5) = \binom{6}{5}(0.68)^5 (0.32)^1$
* $P(X=6) = \binom{6}{6}(0.68)^6 (0.32)^0$

Notice in the table ⇒ $0 \leq P(X=x) < 1$ & $\sum_x P(X=x) = 1$

$\mu = E(X) = 0(0.001) + 1(0.014) + 2(0.073) + \cdots$ لاحظ ان (μ)

$= np = 6 * (0.68) = 4.08$ هو نفسه

COLLEGE

* The Poisson Distribution

* الـ Poisson نفس الـ Binomial بس ما بكون عدد التجارب عدد من المرات
بخليها مفتوحة للـ ∞ يعني على سبيل المثال بدنا نراقب عدد حدوث
حادث على طريق معين بكاميرا وتخيلنا موجودة الكاميرا للـ Infinite
مدى الحياة . الـ Poisson بيراقب معدل حصول حدث معين بمدة زمنية معينة
أو خلال وحدة مساحة ، خلال وحدة حجم ، ... Whatever
خلال فترة معينة = على تكون

* The Poisson distribution is a descrete probability distribution of a random variable x that satisfies these conditions : لازم تحقق الشروط الآتية

1. The experiment consists of counting the number of times x an event occurs in a given interval of time, area, or volume.

2. The prob. of the event occuring is the same for each interval.

3. The no. of occurences in one interval is independent of the number of occurences in other intervals.

$X \sim Poi(\mu)$ يكتب الـ p.d.f بالشكل التالي :

$P(X=x)$ اذا اعرف ان Poisson كيف احسب Prob. :

$$\boxed{P(X=x) = \frac{e^{-\mu}\mu^{x}}{x!}}, \quad x = 0, 1, 2, \ldots \infty$$

يعطينا اياه من الجدول كما في الكتاب $\leftarrow$ Page 241

ميزة : Mean $= E(X) = \mu$, Variance $= \sigma^2 = \mu$

* eg) If $X \sim Poi(3)$, then find $E(X^2)$

sol) $\mu = 3$, $\sigma^2 = 3$, $E(X^2) = \sigma^2 + \mu^2 = 3 + 9 = 12$

* eg) If $X \sim Poi(\mu)$ and $P(X=0) = P(X=1)$ find μ

sol) $P(X=x) = \frac{\mu^{x} \cdot e^{-\mu}}{x!} \Rightarrow P(X=0) = \frac{e^{-\mu} * \mu^0}{0!} = \frac{e^{-\mu} * 1}{1} = e^{-\mu}$

$P(X=1) = \frac{\mu^1 e^{-\mu}}{1!} \longrightarrow \mu e^{-\mu}$

Because $P(X=0) = P(X=1) \Rightarrow \mu e^{-\mu} = e^{-\mu} \longrightarrow \mu = 1$

* يرجى الرجوع الى الكتاب Page 240 $\leftarrow$ EXAMPLE 2 * لحساب Poisson من الجدول Page 673

فرضنا هنا تناسب For a given year find $P(X=4)$ $\Rightarrow$ $3 \longrightarrow 1$ month

$\mu \longrightarrow 12$ month

$P(X=4) = \frac{e^{-36}(36)^4}{4!} = \ldots$ $\leftarrow$ $\mu = 36$

EXAMPLE 6

Finding a Binomial Probability Using a Table

About 10% of workers (ages 16 years and older) in the United States commute to their jobs by carpooling. You randomly select eight workers. What is the probability that exactly four of them carpool to work? Use a table to find the probability. *(Source: American Community Survey)*

SOLUTION

A portion of Table 2 in Appendix B is shown here. Using the distribution for $n = 8$ and $p = 0.1$, you can find the probability that $x = 4$, as shown by the highlighted areas in the table.

		p												
n	***x***	**.01**	**.05**	**.10**	**.15**	**.20**	**.25**	**.30**	**.35**	**.40**	**.45**	**.50**	**.55**	**.60**
2	0	.980	.902	.810	.723	.640	.563	.490	.423	.360	.303	.250	.203	.160
	1	.020	.095	.180	.255	.320	.375	.420	.455	.480	.495	.500	.495	.480
	2	.000	.002	.010	.023	.040	.063	.090	.123	.160	.203	.250	.303	.360
3	0	.970	.857	.729	.614	.512	.422	.343	.275	.216	.166	.125	.091	.064
	1	.029	.135	.243	.325	.384	.422	.441	.444	.432	.408	.375	.334	.288
	2	.000	.007	.027	.057	.096	.141	.189	.239	.288	.334	.375	.408	.432
	3	.000	.000	.001	.003	.008	.016	.027	.043	.064	.091	.125	.166	.216
8	0	.923	.663	.430	.272	.168	.100	.058	.032	.017	.008	.004	.002	.001
	1	.075	.279	.383	.385	.336	.267	.198	.137	.090	.055	.031	.016	.008
	2	.003	.051	.149	.238	.294	.311	.296	.259	.209	.157	.109	.070	.041
	3	.000	.005	.033	.084	.147	.208	.254	.279	.279	.257	.219	.172	.124
	4	.000	.000	.005	.018	.046	.087	.136	.188	.232	.263	.273	.263	.232
	5	.000	.000	.000	.003	.009	.023	.047	.081	.124	.172	.219	.257	.279
	6	.000	.000	.000	.000	.001	.004	.010	.022	.041	.070	.109	.157	.209
	7	.000	.000	.000	.000	.000	.000	.001	.003	.008	.016	.031	.055	.090
	8	.000	.000	.000	.000	.000	.000	.000	.000	.001	.002	.004	.008	.017

According to the table, the probability is 0.005. You can check this result using technology. As shown at the right using Minitab, the probability is

0.0045927.

After rounding to three decimal places, the probability is 0.005, which is the same value found using the table.

MINITAB

Probability Density Function

Binomial with n = 8 and p = 0.1

x	P(X = x)
4	0.0045927

Interpretation So, the probability that exactly four of the eight workers carpool to work is 0.005. Because 0.005 is less than 0.05, this can be considered an unusual event.

TRY IT YOURSELF 6

About 5% of workers (ages 16 years and older) in the United States commute to their jobs by using public transportation (excluding taxicabs). You randomly select six workers. What is the probability that exactly two of them use public transportation to get to work? Use a table to find the probability. *(Source: American Community Survey)*

Answer: Page A35

$\mu = np = 6000 \times \frac{1}{2500} = 2.4$

$P(X=4) = \frac{e^{-2.4}(2.4)^4}{4!} = 0.125$

a) $\boxed{n = 6000}$

$\boxed{p = \frac{1}{2500}}$

$q = \frac{2499}{2500}$

$P(X=4) = \binom{6000}{4}\left(\frac{1}{2500}\right)^4 \left(\frac{2499}{2500}\right)^{6000-4}$

$= 0.125$

Extending Concepts

27. **Comparing Binomial and Poisson Distributions** An automobile manufacturer finds that 1 in every 2500 automobiles produced has a specific manufacturing defect. (a) Use a binomial distribution to find the probability of finding 4 cars with the defect in a random sample of 6000 cars. (b) The Poisson distribution can be used to approximate the binomial distribution for large values of n and small values of p. Repeat part (a) using the Poisson distribution and compare the results.

28. **Hypergeometric Distribution** Binomial experiments require that any sampling be done with replacement because each trial must be independent of the others. The **hypergeometric distribution** also has two outcomes: success and failure. The sampling, however, is done without replacement. For a population of N items having k successes and $N - k$ failures, the probability of selecting a sample of size n that has x successes and $n - x$ failures is given by

$$P(x) = \frac{({}_kC_x)({}_{N-k}C_{n-x})}{{}_NC_n}.$$

In a shipment of 15 microchips, 2 are defective and 13 are not defective. A sample of three microchips is chosen at random. Use the above formula to find the probability that (a) all three microchips are not defective, (b) one